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WEAK CONVERGENCE OF A MILSTEIN SCHEME

AND THE APPROXIMATION OF ITERATED INTEGRALS

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YSSSA24, VÄXJÖ

SETUP

Stochastic Evolution Equation (SEE)

$$dX_t = [AX_t + F(X_t)]\ dt + B(X_t)\, dW_t$$

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$$\mathrm{d}X_t = [AX_t + F(X_t)]\, \mathrm{d}t + B(X_t)\, \mathrm{d}W_t$$

Assumptions

- H, U separable Hilbert spaces
- $A\colon D(A) \subseteq H \to H$ sectorial operator ($\Rightarrow$ analytic semigroup)
- $H_\alpha = (D(A^\alpha), \|(-A)^\alpha \cdot\|_H)$, $H_{-\alpha} = \overline{(H, \|(-A)^{-\alpha} \cdot\|_H)}$
- $F \in \mathrm{Lip}^4(H, H_{-\alpha})$, $\alpha \in [0, 1)$
- $B \in \mathrm{Lip}^4(H, \mathrm{HS}(U, H_{-\beta}))$, $\beta = 0$
- $(W_t)_{t\in[0,T]}$ is an Id_U-cylindrical Wiener process
- $X_0 = \xi \in L^5(\Omega, H)$

SETUP

Stochastic Evolution Equation (SEE)

$$\mathrm{d}X_t = [AX_t + F(X_t)]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t$$

Mild Solution

$$X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s$$

existence and uniqueness are guaranteed under the above assumptions

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Van Neerven, Veraar, and Weis, "Stochastic evolution equations in UMD Banach spaces".

MILD STOCHASTIC CALCULUS

Mild Itô Process

$$X_t = S_{0,t}X_0 + \int_0^t S_{s,t}Y_s \,\mathrm{d}s + \int_0^t S_{s,t}Z_s \,\mathrm{d}W_s$$

where $S\colon \{(s,t) \in [0,T]^2 : s < t\} \to L(H)$ satisfies $S_{r,t}S_{s,r} = S_{s,t}$

Da Prato, Jentzen, and Röckner, "A mild Itô formula for SPDEs".

MILD STOCHASTIC CALCULUS

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Mild Itô Formula for $\varphi\colon H \to V$

$$\varphi(X_t) = \varphi(S_{0,t}X_0) + \int_0^t \varphi'(S_{s,t}X_s) \,\mathrm{d}X_s + \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \varphi''(S_{s,t}X_s)(S_{s,t}Z_se_k, S_{s,t}Z_se_k) \,\mathrm{d}s$$

Da Prato, Jentzen, and Röckner, “A mild Itô formula for SPDEs”.

EXPONENTIAL MILSTEIN SCHEME

Discrete Scheme (with step size $h = T/N$)

$$Z^N_{(n+1)h} = e^{hA}\Big[Z^N_{nh} + hF(Z^N_{nh}) + \int_{nh}^{(n+1)h} B(Z^N_{nh})\,\mathrm{d}W_s + \int_{nh}^{(n+1)h}\int_{nh}^{s} B'(Z^N_{nh})B(Z^N_{nh})\,\mathrm{d}W_u\,\mathrm{d}W_s\Big]$$

Jentzen and Röckner, "A Milstein Scheme for SPDEs".

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Continuous Interpolation (where $\lfloor t \rfloor_h = \max\{k \cdot h \le t,\ k \in \mathbb{Z}\}$)

$$Z^N_t = e^{tA}Z^N_0 + \int_0^t e^{(t-\lfloor s\rfloor_h)A}F(Z^N_{\lfloor s\rfloor_h})\,\mathrm{d}s + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\left[B(Z^N_{\lfloor s\rfloor_h}) + B'(Z^N_{\lfloor s\rfloor_h})\left(\int_{\lfloor s\rfloor_h}^{s} B(Z^N_{\lfloor s\rfloor_h})\,\mathrm{d}W_u\right)\right]\mathrm{d}W_s$$

Jentzen and Röckner, "A Milstein Scheme for SPDEs".

EXPONENTIAL MILSTEIN SCHEME

Continuous Interpolation (where $\lfloor t \rfloor_h = \max\{k \cdot h \leq t,\, k \in \mathbb{Z}\}$)

$$Z_t^N = e^{tA} Z_0^N + \int_0^t e^{(t-\lfloor s \rfloor_h)A} F(Z_{\lfloor s \rfloor_h}^N)\, ds$$
$$+ \int_0^t e^{(t-\lfloor s \rfloor_h)A} \left[B(Z_{\lfloor s \rfloor_h}^N) + B'(Z_{\lfloor s \rfloor_h}^N) \left(\int_{\lfloor s \rfloor_h}^s B(Z_{\lfloor s \rfloor_h}^N)\, dW_u \right) \right] dW_s$$

As Mild Itô process

$$Z_t^N = e^{tA} Z_0^N + \int_0^t e^{(t-s)A} \tilde{F}_s\, ds + \int_0^t e^{(t-s)A} \tilde{B}_s\, dW_s$$

where $\tilde{F}_s = e^{(s-\lfloor s \rfloor_h)A} F$ and $\tilde{B}_s = e^{(s-\lfloor s \rfloor_h)A} [B + \int B'B\, dW]$

Jentzen and Röckner, "A Milstein Scheme for SPDEs".

WEAK VS. STRONG CONVERGENCE

Strong Convergence of Order γ

$$\mathbb{E}\left[\left\|Z_T^N - X_T\right\|_H\right] \leq C \cdot N^{-\gamma}$$

Weak Convergence of Order γ

$$\left\|\mathbb{E}\left[\varphi(Z_T^N) - \varphi(X_T)\right]\right\|_V \leq C \cdot N^{-\gamma}$$

for some class of test functions $\varphi\colon H \to V$

MAIN RESULT

Theorem (Weak Convergence of the Exp. Milstein Scheme)

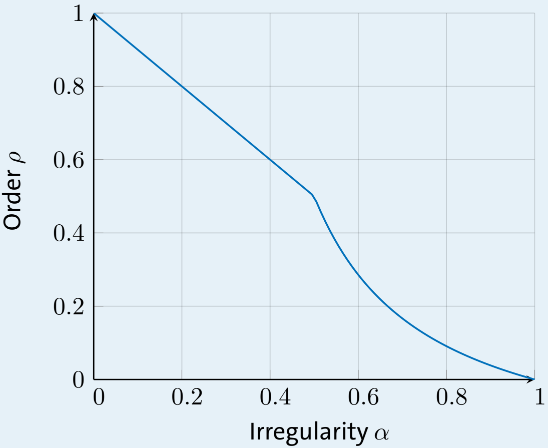
Let $\varphi \in \text{Lip}^4(H, V)$ *and* $\alpha \in [0, 1/2)$. *Then it holds that*

$$\left\| \mathbb{E}[\varphi(Z_T^N) - \varphi(X_T)] \right\|_V \leq C \cdot N^{-(1-\alpha-\varepsilon)} .$$

If $\alpha \in [1/2, 1)$ *then it holds*

$$\left\| \mathbb{E}[\varphi(Z_T^N) - \varphi(X_T)] \right\|_V \leq C \cdot N^{-\left(\frac{1-\alpha}{4\alpha-1}-\varepsilon\right)} .$$

MAIN RESULT



REGULARITY PROPERTIES FOR KBE

SEE: $dX_t^x = \left[AX_t^x + F(X_t^x)\right]\, dt + B(X_t^x)\, dW_t, \quad X_0^x = x$

Kolmogorov Backward Equation (KBE)

- $F \in \mathrm{Lip}^4(H, H_1)$
- $B \in \mathrm{Lip}^4(H, \mathrm{HS}(U, H_1))$
- $\varphi \in \mathrm{Lip}^4(H, V)$

$u\colon [0,T] \times H \to V, \quad (t,x) \mapsto u(t,x) := \mathbb{E}\left[\varphi(X_{T-t}^x)\right]$

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = -\frac{\partial}{\partial x} u(t,x)\,(Ax + F(x)) \\ \qquad\qquad - \frac{1}{2}\sum_{n\in\mathbb{N}} \frac{\partial^2}{\partial x^2} u(t,x)\,(B(x)e_n, B(x)e_n) & t \in [0,T),\ x \in H_1\,, \\ u(T,x) = \varphi(x) & x \in H \end{cases}$$

REGULARITY PROPERTIES FOR KBE

for $k \leq 4$, $\delta_i \in [0, \frac{1}{2})$, $\sum_{i=1}^{k} \delta_i < \frac{1}{2}$, $t \in [0, T)$, $x \in H$, $v_i \in H$

$$\left\| \frac{\partial^k}{\partial x^k} u(t,x)(v_1, \ldots, v_k) \right\|_V \leq c_{\delta_1,\ldots,\delta_k} \cdot (T-t)^{-\sum_{i=1}^{k} \delta_i} \prod_{i=1}^{k} \|v_i\|_{H_{-\delta_i}}$$

Andersson et al., "Regularity Properties for Solutions of Infinite Dimensional Kolmogorov Equations in Hilbert Spaces".

PROOF

Mollified Equation

$$\begin{cases} \mathrm{d}X_t^{\varepsilon,\delta} = \left[AX_t^{\varepsilon,\delta} + e^{\varepsilon A}F(X_t^{\varepsilon,\delta})\right] \mathrm{d}t + e^{\varepsilon A}B(X_t^{\varepsilon,\delta})\, \mathrm{d}W_t, \quad t \in (0,T] \\ X_0^{\varepsilon,\delta} = e^{\delta A}\xi \end{cases}$$

$$\begin{aligned} \|\mathbb{E}[\varphi(Z_T) - \varphi(X_T)]\| \leq \lim_{\delta\to 0} \sup_{\varepsilon\in(0,T]} & \|\mathbb{E}[\varphi(X_T) - \varphi(X_T^{0,\delta})]\| \\ & + \|\mathbb{E}[\varphi(X_T^{0,\delta}) - \varphi(X_T^{\varepsilon,\delta})]\| \\ & + \|\mathbb{E}[\varphi(X_T^{\varepsilon,\delta}) - \varphi(Z_T^{\varepsilon,\delta})]\| \\ & + \|\mathbb{E}[\varphi(Z_T^{\varepsilon,\delta}) - \varphi(Z_T^{0,\delta})]\| \\ & + \|\mathbb{E}[\varphi(Z_T^{0,\delta}) - \varphi(Z_T)]\| \end{aligned}$$

PROOF (REGULAR CASE)

Remember:

$$Z_t = e^{tA} Z_0 + \int_0^t e^{(t-\lfloor s \rfloor_h)A} F(Z_{\lfloor s \rfloor_h})\, \mathrm{d}s + \int_0^t e^{(t-\lfloor s \rfloor_h)A} \tilde{B}_s\, \mathrm{d}W_s$$

where $\tilde{B} = B + \int B'B \, \mathrm{d}W$

Step 1: Introduce auxiliary process $\bar{Z}$

$$\bar{Z}_t = e^{tA} Z_0 + \int_0^t e^{(t-s)A} F(Z_{\lfloor s \rfloor_h})\, \mathrm{d}s + \int_0^t e^{(t-s)A} \tilde{B}_s\, \mathrm{d}W_s$$

$\Rightarrow \bar{Z}$ is a strong solution of $\mathrm{d}\bar{Z}_t = \left[A\bar{Z}_t + F(Z_{\lfloor t \rfloor_h}) \right] \mathrm{d}t + \tilde{B}_t \, \mathrm{d}W_t$

Jentzen and Kurniawan, "Weak Convergence Rates for Euler-Type Approximations of Semilinear Stochastic Evolution Equations with Nonlinear Diffusion Coefficients".

PROOF (REGULAR CASE)

Step 2: Use the standard Itô formula for $\bar{Z}_t$

$$\mathbb{E}\left[\varphi(\bar{Z}_T) - \varphi(X_T)\right] = \mathbb{E}\left[u(T, \bar{Z}_T) - u(0, \bar{Z}_0)\right]$$

$$\overset{\text{Itô}}{=} \mathbb{E}\left[\int_0^T \frac{\partial}{\partial t} u(t, \bar{Z}_t)\, \mathrm{d}t + \int_0^T \frac{\partial}{\partial x} u(t, \bar{Z}_t) \left(A\bar{Z}_t + F(Z_{\lfloor t \rfloor_h})\right) \mathrm{d}t\right]$$

$$+ \mathbb{E}\left[\frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^T \frac{\partial^2}{\partial x^2} u(t, \bar{Z}_t) \left(\tilde{B}_t e_k, \tilde{B}_t e_k\right) \mathrm{d}t\right]$$

PROOF (REGULAR CASE)

Step 3: Use the KBE for X_t

$$\mathbb{E}\left[\varphi(\bar{Z}_T) - \varphi(X_T)\right] = \mathbb{E}\left[u(T, \bar{Z}_T) - u(0, \bar{Z}_0)\right]$$

$$\overset{\text{KBE}}{=} \mathbb{E}\left[\int_0^T \frac{\partial}{\partial x} u(t, \bar{Z}_t)\left(F(Z_{\lfloor t \rfloor})\right) - \frac{\partial}{\partial x} u(t, \bar{Z}_t)\left(F(\bar{Z}_t)\right)\, \mathrm{d}t\right]$$

$$+ \mathbb{E}\left[\frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^T \frac{\partial^2}{\partial x^2} u(t, \bar{Z}_t)\left(\tilde{B}_t e_k, \tilde{B}_t e_k\right) - \frac{\partial^2}{\partial x^2} u(t, \bar{Z}_t)\left(B(\bar{Z}_t) e_k, B(\bar{Z}_t) e_k\right)\, \mathrm{d}t\right]$$

PROOF (REGULAR CASE)

Step 4: Use

- mild Itô formula,
- the fundamental theorem of calculus,
- the regularity properties of the KBE
- and nice properties of analytic semigroups

to estimate all terms.

CONCLUSIONS

Results

- same order as in the finite-dimensional case
- same (weak) order as exponential Euler scheme →MLMC
- also holds for certain variants, e.g. linear-implicit Milstein scheme

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Results

- same order as in the finite-dimensional case
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Open questions/problems

- order $1 - \alpha$ for all $\alpha \in [0, 1)$
- what about $\beta \neq 0$?
- different schemes, e.g. derivative-free or Runge–Kutta schemes
- SEEs in Banach spaces

ITERATED STOCHASTIC INTEGRALS

Milstein term

$$\int_{nh}^{(n+1)h} B'(Z_{nh}) \left(\int_{nh}^{s} B(Z_{nh}) \, \mathrm{d}W_u \right) \mathrm{d}W_s$$

In finite dimensions (kth component)

$$\sum_{i=1}^{m} \int_{nh}^{(n+1)h} \left[\sum_{l=1}^{d} \frac{\partial B^{k,i}(Z_{nh})}{\partial x^l} \left(\sum_{j=1}^{m} \int_{nh}^{s} B^{l,j}(Z_{nh}) \, \mathrm{d}W_u^j \right) \right] \mathrm{d}W_s^i$$

$$= \sum_{i,j=1}^{m} \sum_{l=1}^{d} \frac{\partial B^{k,i}(Z_{nh})}{\partial x^l} B^{l,j}(Z_{nh}) \cdot \int_{nh}^{(n+1)h} \int_{nh}^{s} \mathrm{d}W_u^j \, \mathrm{d}W_s^i$$

Diagonal terms are easy:

$$\int_0^h \int_0^s \mathrm{d}W_u^i \, \mathrm{d}W_s^i = \frac{1}{2}(W_h^i)^2 - \frac{1}{2}h$$

We need to simulate for $i \neq j$

$$I_{i,j}(h) = \int_0^h \int_0^s \mathrm{d}W_u^i \, \mathrm{d}W_s^j$$

There are different algorithms:

- Milstein (1988)
- Wiktorsson (2001)
- Mrongowius, Rößler (2022)

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→ `LevyArea.jl`

- fast implementations in Julia
- easy to use
- over 50.000 downloads

Kastner and Rößler, “An analysis of approximation algorithms for iterated stochastic integrals and a Julia and Matlab simulation toolbox”.

In infinite dimensions

- $(W_t)_{t\geq 0}$ Q-Wiener process, $\operatorname{tr} Q < \infty$
- $(W_t^K)_{t\geq 0} = \sum_{j\in\mathcal{J}_K} \langle W_t, e_j\rangle e_j$ finite-dimensional projection

$$\mathcal{E}_p(K,D) = \left\| \int_0^h \Psi\left(\int_0^s \Phi \, \mathrm{d}W_u^K\right) \mathrm{d}W_s^K - \sum_{i,j\in\mathcal{J}_K} \hat{I}_{i,j}^{(D)}(h)\Psi(\Phi e_i)e_j \right\|_{L^p(\Omega,H)}$$

Leonhard & Rößler showed

$$\mathcal{E}_2(K,D) \leq C \cdot (\max_{j\in\mathcal{J}_K} \eta_j)\sqrt{K^2(K-1)} \cdot \frac{h}{D}$$

$$\mathcal{E}_2(K,D) \leq C \cdot \frac{(\max_{j\in\mathcal{J}_K} \eta_j)^{\frac{1}{2}}}{\min_{j\in\mathcal{J}_K} \eta_j}\sqrt{(\operatorname{tr} Q)^3} \cdot \frac{h}{D}$$

It's possible to improve this to (unpublished)

$$\mathcal{E}_2(K,D) \leq C \cdot \sqrt{(\operatorname{tr} Q)^2 - \operatorname{tr} Q^2} \cdot \sqrt{K} \cdot \frac{h}{D}$$

$$\mathcal{E}_p(K,D) \leq C \cdot ((\operatorname{tr} Q^{\frac{1}{2}})^2 - \operatorname{tr} Q) \cdot \sqrt{K} \cdot \frac{h}{D}$$

Leonhard and Rößler, "Iterated stochastic integrals in infinite dimensions: approximation and error estimates".

CONCLUSIONS

- iterated integrals are important
  - for numerics (higher order approximation schemes)
  - but also interesting on their own (central object in rough path theory)
- many estimates can still be improved
- there are currently no $L^p(\Omega)$ estimates
- cylindrical Wiener processes ($\operatorname{tr} Q = \infty$) are difficult
- what about Banach space-valued integrands?

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