Weak Convergence of a Milstein Scheme for SPDEs

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Setup

Stochastic Partial Differential Equation

$$\mathrm{d}X_t = [AX_t + F(X_t)]\ \mathrm{d}t + B(X_t)\,\mathrm{d}W_t$$

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- H, U Hilbert spaces
- $A\colon D(A) \subseteq H \to H$ generates an analytic semigroup with negative growth bound, equivalently $\sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\} < 0$
- $F \in \mathrm{Lip}^4(H, H)$
- $B \in \mathrm{Lip}^4(H, \mathrm{HS}(U, H))$
- $(W_t)_{t\in[0,T]}$ is an Id_U-cylindrical Wiener process
- $X_0 = x \in H$

Setup

Stochastic Partial Differential Equation

$$\mathrm{d}X_t = [AX_t + F(X_t)]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t$$

Mild Solution

$$X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}F(X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s)\,\mathrm{d}W_s$$

existence and uniqueness are guaranteed under the above assumptions

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Neerven, Veraar, and Weis, "Stochastic evolution equations in UMD Banach spaces".

Mild Stochastic Calculus

Mild Itô Process

$$X_t = S_{t_0,t} X_{t_0} + \int_{t_0}^{t} S_{s,t} Y_s \, \mathrm{d}s + \int_{t_0}^{t} S_{s,t} Z_s \, \mathrm{d}W_s$$

where $S \colon \{(s,t) \in [0,T]^2 \colon s < t\} \to L(H)$ satisfies $S_{r,t} S_{s,r} = S_{s,t}$

Da Prato, Jentzen, and Röckner, "A mild Itô formula for SPDEs".

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Mild Itô Formula

$$\varphi(X_t) = \varphi(S_{t_0,t} X_{t_0}) + \int_{t_0}^{t} \varphi'(S_{s,t} X_s) S_{s,t} Y_s \,\mathrm{d}s + \int_{t_0}^{t} \varphi'(S_{s,t} X_s) S_{s,t} Z_s \,\mathrm{d}W_s + \frac{1}{2} \int_{t_0}^{t} \sum_{u \in \mathbb{U}} \varphi''(S_{s,t} X_s)(S_{s,t} Z_s u, S_{s,t} Z_s u) \,\mathrm{d}s$$

where $\varphi\colon H \to V$ and $\mathbb{U}$ is an ONB of U

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Exponential Milstein Scheme

equidistant time-discretization with step size h and $\lfloor t \rfloor_h = \max\{k \cdot h \leq t,\, k \in \mathbb{Z}\}$:

$$Z_{(n+1)h} = e^{hA}\left[Z_{nh} + hF(Z_{nh}) + \int_{nh}^{(n+1)h} B(Z_{nh}) + B'(Z_{nh})\left(\int_{nh}^{s} B(Z_{nh})\,\mathrm{d}W_u\right)\,\mathrm{d}W_s\right]$$

Jentzen and Röckner, "A Milstein Scheme for SPDEs".

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$$\begin{aligned} Z_t = e^{tA}Z_0 &+ \int_0^t e^{(t-\lfloor s\rfloor_h)A}F(Z_{\lfloor s\rfloor_h})\,\mathrm{d}s \\ &+ \int_0^t e^{(t-\lfloor s\rfloor_h)A}\left[B(Z_{\lfloor s\rfloor_h}) + B'(Z_{\lfloor s\rfloor_h})\left(\int_{\lfloor s\rfloor_h}^{s} B(Z_{\lfloor s\rfloor_h})\,\mathrm{d}W_u\right)\right]\,\mathrm{d}W_s \end{aligned}$$

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$$Z_t = e^{tA} Z_0 + \int_0^t e^{(t-s)A} \underbrace{e^{(s-\lfloor s \rfloor_h)A} F(Z_{\lfloor s \rfloor_h})}_{\text{mild drift}} \, \mathrm{d}s + \int_0^t e^{(t-s)A} \underbrace{e^{(s-\lfloor s \rfloor_h)A} \left[B(Z_{\lfloor s \rfloor_h}) + B'(Z_{\lfloor s \rfloor_h}) \left(\int_{\lfloor s \rfloor_h}^{s} B(Z_{\lfloor s \rfloor_h}) \, \mathrm{d}W_u \right) \right]}_{\text{mild diffusion}} \mathrm{d}W_s$$

Thus, Z_t is a mild Itô process!

Jentzen and Röckner, "A Milstein Scheme for SPDEs".

Regularity Properties for KBE

Kolmogorov Backward Equation

$$\varphi \in \mathrm{Lip}^4(H, V)$$

$$u\colon [0, T] \times H \to V, \quad (t, x) \mapsto \mathbb{E}\left[\varphi(X_{T-t}^x)\right]$$

$$\frac{\partial}{\partial t} u(t, x) = -\frac{\partial}{\partial x} u(t, x)\,(Ax + F(x)) - \frac{1}{2} \sum_{u \in \mathbb{U}} \frac{\partial^2}{\partial x^2} u(t, x)\,(B(x)u, B(x)u)$$

$$u(T, x) = \varphi(x)$$

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$$u(T,x) = \varphi(x)$$

$$\left\| \frac{\partial^k}{\partial x^k} u(t,x) \right\|_{L^k(H,V)} \leq C \quad \forall (t,x) \in [0,T) \times H, k \in \{0, \dots, 4\}$$

Andersson et al., "Regularity Properties for Solutions of Infinite Dimensional Kolmogorov Equations in Hilbert Spaces".

Main Result

Theorem (Weak convergence of the exponential Milstein scheme)

Let $\varphi \in \mathrm{Lip}^4(H, V)$ *and* $\rho \in [0,1)$. *Then it holds that*

$$\|\mathbb{E}\left[\varphi(Z_t) - \varphi(X_t)\right]\|_V \le C \cdot h^\rho \,.$$

where C *depends on everything but* h.

Proof Sketch

Remember:

$$Z_t = e^{tA} Z_0 + \int_0^t e^{(t-\lfloor s \rfloor_h)A} F(Z_{\lfloor s \rfloor_h}) \,\mathrm{d}s + \int_0^t e^{(t-\lfloor s \rfloor_h)A} \tilde{B}_s \,\mathrm{d}W_s$$

where $\tilde{B}_t = B(Z_{\lfloor t \rfloor_h}) + B'(Z_{\lfloor t \rfloor_h}) \left(\int_{\lfloor t \rfloor_h}^t B(Z_{\lfloor t \rfloor_h}) \,\mathrm{d}W_u \right)$

Step 1: Introduce appropriate process $\bar{Z}_t$ and use triangle inequality

$$\bar{Z}_t = e^{tA} Z_0 + \int_0^t e^{(t-s)A} F(Z_{\lfloor s \rfloor_h}) \,\mathrm{d}s + \int_0^t e^{(t-s)A} \tilde{B}_s \,\mathrm{d}W_s$$

$\bar{Z}_t$ is a strong solution of $\mathrm{d}\bar{Z}_t = \left[A\bar{Z}_t + F(Z_{\lfloor t \rfloor_h}) \right] \mathrm{d}t + \tilde{B}_t \,\mathrm{d}W_t$

Jentzen and Kurniawan, "Weak Convergence Rates for Euler-Type Approximations of Semilinear Stochastic Evolution Equations with Nonlinear Diffusion Coefficients".

Proof Sketch

Step 2: Use the standard Itô formula for $\bar{Z}_t$

$$\mathbb{E}\Big[\varphi(\bar{Z}_T) - \varphi(X_T)\Big] = \mathbb{E}\Big[u(T, \bar{Z}_T) - u(0, \bar{Z}_0)\Big]$$

$$\overset{\text{Itô}}{=} \mathbb{E}\left[\int_0^T u_{1,0}(t, \bar{Z}_t)\,\mathrm{d}t + \int_0^T u_{0,1}(t, \bar{Z}_t)\left(A\bar{Z}_t + F(Z_{\lfloor t \rfloor_h})\right)\,\mathrm{d}t\right]$$

$$+ \mathbb{E}\left[\frac{1}{2}\sum_{u \in \mathbb{U}} \int_0^T u_{0,2}(t, \bar{Z}_t)\left(\tilde{B}_t u, \tilde{B}_t u\right)\,\mathrm{d}t\right]$$

where $\tilde{B}_t = B(Z_{\lfloor t \rfloor_h}) + B'(Z_{\lfloor t \rfloor_h})\left(\int_{\lfloor t \rfloor_h}^t B(Z_{\lfloor t \rfloor_h})\,\mathrm{d}W_u\right)$

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Proof Sketch

Step 3: Use the KBE for X_t

$$\mathbb{E}\Big[\varphi(\bar{Z}_T) - \varphi(X_T)\Big] = \mathbb{E}\Big[u(T, \bar{Z}_T) - u(0, \bar{Z}_0)\Big]$$

$$\stackrel{\text{KBE}}{=} \mathbb{E}\left[\int_0^T u_{0,1}(t, \bar{Z}_t)\left(F(Z_{\lfloor t \rfloor})\right) - u_{0,1}(t, \bar{Z}_t)\left(F(\bar{Z}_t)\right)\,\mathrm{d}t\right]$$

$$+\,\mathbb{E}\left[\frac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T u_{0,2}(t, \bar{Z}_t)\left(\tilde{B}_t u, \tilde{B}_t u\right) - u_{0,2}(t, \bar{Z}_t)\left(B(\bar{Z}_t)u, B(\bar{Z}_t)u\right)\,\mathrm{d}t\right]$$

Jentzen and Kurniawan, "Weak Convergence Rates for Euler-Type Approximations of Semilinear Stochastic Evolution Equations with Nonlinear Diffusion Coefficients".

Proof Sketch

Step 4: Use the mild Itô formula and the fundamental theorem of calculus to estimate all the terms.

Conclusions

- same order as in the finite-dimensional case
- same order of convergence as exponential Euler scheme $\rightarrow$MLMC

Open questions:

- Do simulations agree with the theory?
- nonlinearities with values in interpolation spaces H_r
- different schemes, e.g. derivative-free Runge–Kutta schemes
- SPDEs in Banach spaces

References

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In Interpolation Spaces

$$\mathrm{d}X_t = [AX_t + F(X_t)]\, \mathrm{d}t + B(X_t)\, \mathrm{d}W_t$$

- $A\colon D(A) \subseteq H \to H$ generates an analytic semigroup with negative growth bound, equivalently $\sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\} < 0$
- $H_r = D((-A)^r)$
- $F \in \mathrm{Lip}^4(H, H_{-\alpha})$, $\alpha \in [0, 1)$
- $B \in \mathrm{Lip}^4(H, \mathrm{HS}(U, H_{-\beta}))$, $\beta \in [0, \frac{1}{2})$
- $(W_t)_{t\in[0,T]}$ is a Id_U-cylindrical Wiener process
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In Interpolation Spaces

Let $\delta_i \in [0, \frac{1}{2})$ and $\sum_{i=1}^{k} \delta_i < \frac{1}{2}$, then we have

$$\|\frac{\partial^k}{\partial x^k} u(t,x)\|_{L(H_{-\delta_1} \times \ldots \times H_{-\delta_k}, V)} \leq C \cdot (T-t)^{-\sum_{i=1}^{k} \delta_i}$$

for all $(t,x) \in [0,T) \times H$.

Andersson et al., "Regularity Properties for Solutions of Infinite Dimensional Kolmogorov Equations in Hilbert Spaces".