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THE ITERATED INTEGRAL EXPANSION

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YSSSA25, LINNAEUS UNIVERSITY, VÄXJÖ

SMOOTH PATHS

$x\colon [0,T] \to \mathbb{R}$ Lipschitz and $f : \mathbb{R} \to \mathbb{R}$ smooth

$$f(x_t) = f(x_0) + \int_0^t f'(x_s)\, \mathrm{d}x_s$$

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&\vdots \\
&= f(x_0) + f'(x_0) \cdot \int_0^t \mathrm{d}x_s + f''(x_0) \cdot \int_0^t \int_0^s \mathrm{d}x_r\,\mathrm{d}x_s + \cdots + R_n
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$$R_n = \int_0^t \cdots \int_0^{s_n} f^{(n+1)}(x_{s_{n+1}})\,\mathrm{d}x_{s_{n+1}} \cdots \mathrm{d}x_{s_1} \in \mathcal{O}(t^{n+1})$$

EXAMPLE 1

$x_t = t$:

$$f(t) = f(0) + f'(0) \cdot \int_0^t \mathrm{d}s + f''(0) \cdot \int_0^t \int_0^s \mathrm{d}r\,\mathrm{d}s + \cdots + R$$

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⇒ this is the (deterministic) Taylor expansion

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$\Rightarrow$ useful for approximating x given the iterated integrals of y

MULTIDIMENSIONAL PATHS

$x : [0, T] \to \mathbb{R}^n$ Lipschitz and $f : \mathbb{R}^n \to \mathbb{R}^m$ smooth

$$f(x_t) = f(x_0) + \partial_i f(x_0) \cdot \int_0^t \mathrm{d}x_s^i + \partial_i \partial_j f(x_0) \cdot \int_0^t \int_0^s \mathrm{d}x_r^i \, \mathrm{d}x_s^j + \cdots + R_n$$

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$$R_n \in \mathcal{O}(t^{n+1})$$

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the (twice) iterated integral only makes sense for $\gamma > \frac{1}{2}$
$\Rightarrow$ Consider the collection $\mathbf{x} = (x, I_{i,j}, I_{i,j,k}, \ldots, I_{i_1,\ldots,i_N})$ for $N = \lfloor \frac{1}{\gamma} \rfloor$ as given

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$$R_n \in \mathcal{O}(t^{\gamma(n+1)}), \quad n \geq N$$

THE STOCHASTIC CASE

From Itôs lemma

$$\varphi(X_{t+h}) = \varphi(X_t) + \int_t^{t+h} \varphi'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_t^{t+h} \varphi''(X_s)\,\mathrm{d}[X]_s$$

we get

$$\begin{aligned} X_{t+h} &= \sum_{\alpha\in\mathcal{A}} f_\alpha(X_t)\cdot I_\alpha(h) + \sum_{\alpha\in\mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(X_\cdot)]_{t,t+h} \\ &= X_t + a(X_t)\cdot I_0(h) + b(X_t)\cdot I_1(h) + b(X_t)b'(X_t)\cdot I_{1,1}(h) + \cdots \end{aligned}$$

for

$$X_t = X_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s$$

HIERARCHICAL SETS

- $\mathcal{A} \subseteq \left\{(j_1, \ldots, j_l) \in \{0, 1, \ldots, d\}^l, l \in \mathbb{N}\right\} \cup \{\nu\}$
- $\mathcal{A} \neq \emptyset$
- $\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty$
- $-\alpha := (j_2, \ldots, j_l) \in \mathcal{A} \quad \forall \alpha \in \mathcal{A} \setminus \{\nu\}$

Examples: $\{\nu, (0), (1)\}$, $\{\nu, (0), (1), (1, 1)\}$

Remainder Set of $\mathcal{A}$: $\mathcal{B}(\mathcal{A}) := \{\alpha \notin \mathcal{A} : -\alpha \in \mathcal{A}\}$

COEFFICIENT FUNCTIONS

$$f_\alpha = \begin{cases} f & l(\alpha) = 0 \\ L^{j_1} f_{-\alpha} & l(\alpha) \geq 1 \end{cases}$$

where

$$L^0 = \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} b^{i,k} b^{j,k} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$L^j = \sum_{i=1}^{d} b^{i,j} \frac{\partial}{\partial x^i}$$

WHY DO WE CARE ABOUT TAYLOR EXPANSIONS?

- important tool in (stochastic) calculus
- heuristic simplifications by focusing on first few terms ("linearisation", "Taylor polynomials")
- important tool in numerics for (stochastic) differential equations
- construct approximation schemes for solutions of (stochastic) differential equations

ITERATED STOCHASTIC INTEGRALS

Milstein term

$$\int_{nh}^{(n+1)h} B'(Z_{nh}) \left(\int_{nh}^{s} B(Z_{nh})\, \mathrm{d}W_u \right) \mathrm{d}W_s$$

In finite dimensions (kth component)

$$\sum_{i=1}^{m} \int_{nh}^{(n+1)h} \left[\sum_{l=1}^{d} \frac{\partial B^{k,i}(Z_{nh})}{\partial x^l} \left(\sum_{j=1}^{m} \int_{nh}^{s} B^{l,j}(Z_{nh})\, \mathrm{d}W_u^j \right) \right] \mathrm{d}W_s^i$$

$$= \sum_{i,j=1}^{m} \sum_{l=1}^{d} \frac{\partial B^{k,i}(Z_{nh})}{\partial x^l} B^{l,j}(Z_{nh}) \cdot \int_{nh}^{(n+1)h} \int_{nh}^{s} \mathrm{d}W_u^j\, \mathrm{d}W_s^i$$

HOW TO SIMULATE ISI'S

Diagonal terms are easy:

$$\int_0^h \int_0^s \mathrm{d}W_u^i \, \mathrm{d}W_s^i = \frac{1}{2}(W_h^i)^2 - \frac{1}{2}h$$

We need to simulate for $i \neq j$

$$I_{i,j}(h) = \int_0^h \int_0^s \mathrm{d}W_u^i \, \mathrm{d}W_s^j$$

There are different algorithms:

- Fourier
- Milstein (1988)
- Wiktorsson (2001)
- Mrongowius, Rößler (2022)

HOW CAN I USE THESE ALGORITHMS?

> *In the future, we foresee that the use of area integrals when simulating strong solutions to SDEs will become as automatic as the use of random numbers from a normal distribution is today. After all, once a good routine has been developed and implemented in numerical libraries, the ordinary user will only need to call this routine from each program and will not need to be concerned with the details of how the routine works.*
>
> — *J. G. Gaines and T. J. Lyons (1994)*

HOW CAN I USE THESE ALGORITHMS?

LevyArea.jl

- fast implementations in Julia
- easy to use
- flexible (different error criteria, correlated noise)
- employs state of the art estimates
- over 60.000 downloads
- MATLAB version also available

HOW TO SIMULATE ISI'S

Idea: Expand the Wiener process into a Fourier series

$$W_t^i = \frac{t}{h} W_h^i + \frac{1}{2} a_0^i + \sum_{r=1}^{\infty} \left(a_r^i \cos\left(\frac{\tau r}{h} t\right) + b_r^i \sin\left(\frac{\tau r}{h} t\right)\right)$$

with a_r^i, b_r^i Gaussian.

This leads to

$$I_{i,j}(h) = \frac{1}{2} W_h^i W_h^j + \sum_{r=1}^{\infty} \left(W_h^i a_r^j - a_r^i W_h^j \right) + \frac{\tau}{2} \sum_{r=1}^{\infty} r \left(a_r^i b_r^j - b_r^i a_r^j \right)$$

IN INFINITE DIMENSIONS

- $(W_t)_{t\geq 0}$ Q-Wiener process, $\operatorname{tr} Q < \infty$
- $(W_t^K)_{t\geq 0} = \sum_{j\in \mathcal{J}_K} \langle W_t, e_j\rangle e_j$ finite-dimensional projection

$$\mathcal{E}_p(K,D) = \left\| \int_0^h \Psi \left(\int_0^s \Phi \, \mathrm{d}W_u^K \right) \mathrm{d}W_s^K - \sum_{i,j\in \mathcal{J}_K} \hat{I}_{i,j}^{(D)}(h)\Psi(\Phi e_i)e_j \right\|_{L^p(\Omega,H)}$$

IN INFINITE DIMENSIONS

Leonhard & Rößler showed

$$\mathcal{E}_2(K,D) \le C \cdot (\max_{j\in\mathcal{J}_K} \eta_j)\sqrt{K^2(K-1)} \cdot \frac{h}{D}$$

$$\mathcal{E}_2(K,D) \le C \cdot \frac{(\max_{j\in\mathcal{J}_K} \eta_j)^{\frac{1}{2}}}{\min_{j\in\mathcal{J}_K} \eta_j}\sqrt{(\operatorname{tr} Q)^3} \cdot \frac{h}{D}$$

It’s possible to improve this to (unpublished)

$$\mathcal{E}_2(K,D) \le C \cdot \sqrt{(\operatorname{tr} Q)^2 - \operatorname{tr} Q^2} \cdot \sqrt{K} \cdot \frac{h}{D}$$

$$\mathcal{E}_p(K,D) \le C_p \cdot ((\operatorname{tr} Q^{\frac{1}{2}})^2 - \operatorname{tr} Q) \cdot \sqrt{K} \cdot \frac{h}{D}$$

CONCLUSIONS

- iterated integrals are important
  - for numerics (higher order approximation schemes)
  - but also interesting on their own (central object in rough path theory)
- many estimates can still be improved
- there are currently no $L^p(\Omega)$ estimates
- cylindrical Wiener processes ($\operatorname{tr} Q = \infty$) are difficult
- what about Banach space-valued integrands?

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