Iterated Integrals

and how to simulate them

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Overview

1. What are we talking about?
2. Why do we care?
3. How do we do it *in theory*?
4. How do we do it *in practice*?

Definition (deterministic)

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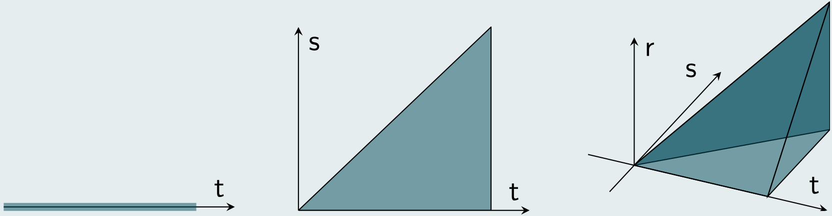

$$\int_0^1 1\,\mathrm{d}t \qquad \int_0^1\int_0^t 1\,\mathrm{d}s\,\mathrm{d}t \qquad \int_0^1\int_0^t\int_0^s 1\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t$$

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$$p_0(x) = 1 \quad p_1(x) = x \quad p_2(x) = \frac{1}{2}x^2 \quad \ldots \quad p_n(x) = \frac{1}{n!}x^n$$

Definition (stochastic)

Iterated stochastic integrals

$$I_i(h) = \int_0^h \mathrm{d}W_t^i$$

$$I_{i,j}(h) = \int_0^h \int_0^t \mathrm{d}W_s^i \, \mathrm{d}W_t^j$$

$$I_{i,j,k}(h) = \int_0^h \int_0^t \int_0^s \mathrm{d}W_r^i \, \mathrm{d}W_s^j \, \mathrm{d}W_t^k$$

$$\vdots$$

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Definition (stochastic)

Twice iterated stochastic integrals

$$I_i(h) = \int_0^h \mathrm{d}W_t^i = W_h^i \qquad \text{simple}$$

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$$\vdots$$

Taylor Expansions (deterministic)

From the FTC

$$f(x+h) = f(x) + \int_x^{x+h} f'(t)\,\mathrm{d}t$$

we get

$$\begin{aligned} f(x+h) &= \sum_{k=0}^{r} f^{(k)}(x) \cdot p_k(h) + R_{r+1}(x, x+h) \\ &= f(x) + f'(x) \cdot \frac{h}{1!} + f''(x) \cdot \frac{h^2}{2!} + \cdots + f^{(r)}(x) \cdot \frac{h^r}{r!} \\ &+ \int_x^{x+h} \cdots \int_x^{s_2} f^{(r+1)}(s_1)\,\mathrm{d}s_1 \cdots \mathrm{d}s_{r+1} \end{aligned}$$

Taylor Expansions (stochastic)

From Itôs lemma

$$\varphi(X_{t+h}) = \varphi(X_t) + \int_t^{t+h} \varphi'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_t^{t+h} \varphi''(X_s)\,\mathrm{d}[X]_s$$

we get

$$\begin{aligned} X_{t+h} &= \sum_{\alpha\in\mathcal{A}} f_\alpha(X_t)\cdot I_\alpha(h) + \sum_{\alpha\in\mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(X_\cdot)]_{t,t+h} \\ &= X_t + a(X_t)\cdot I_0(h) + b(X_t)\cdot I_1(h) + b(X_t)b'(X_t)\cdot I_{1,1}(h) + \cdots \end{aligned}$$

for

$$X_t = X_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s$$

Why Do We Care About Taylor Expansions?

- important tool in (stochastic) calculus
- heuristic simplifications by focusing on first few terms ("linearisation", "Taylor polynomials")
- important tool in numerics for (stochastic) differential equations
- construct approximation schemes for solutions of (stochastic) differential equations

How to Simulate Iterated Integrals?

Diagonal terms are easy:

$$I_{i,i}(h) = \int_0^h \int_0^s \mathrm{d}W_u^i \,\mathrm{d}W_s^i = \frac{1}{2}(W_h^i)^2 - \frac{1}{2}h$$

We need to simulate for $i \neq j$

$$I_{i,j}(h) = \int_0^h \int_0^s \mathrm{d}W_u^i \,\mathrm{d}W_s^j$$

There are different algorithms:

- Fourier
- Milstein (1988)
- Wiktorsson (2001)
- Mrongowius, Rößler (2022)

How to Simulate Iterated Integrals?

Idea: Expand the Wiener process into a Fourier series

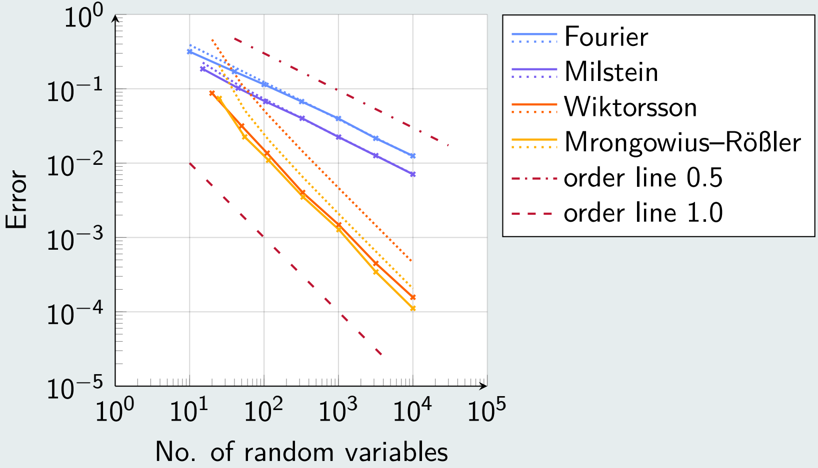
$$W_t^i = \frac{t}{h} W_h^i + \frac{1}{2} a_0^i + \sum_{r=1}^{\infty} \left(a_r^i \cos\left(\frac{\tau r}{h} t\right) + b_r^i \sin\left(\frac{\tau r}{h} t\right)\right)$$

with a_r^i, b_r^i Gaussian.

This leads to

$$I_{i,j}(h) = \frac{1}{2} W_h^i W_h^j + \sum_{r=1}^{\infty} \left(W_h^i a_r^j - a_r^i W_h^j \right) + \frac{\tau}{2} \sum_{r=1}^{\infty} r \left(a_r^i b_r^j - b_r^i a_r^j \right)$$

Comparison of Approximation Algorithms



Effective Order

Error e w.r.t. required number of elementary operations c

$$e \in O(c^{-p_{\text{eff}}})$$

SDE Scheme	Order	Order of Iter. Int. Approx.	Effective Order
Euler–Maruyama	$^1/_2$	–	$^1/_2$
Milstein	1	$^1/_2$	$^1/_2$
Milstein	1	1	$^2/_3$
Milstein	1	α	$^{2\alpha}/_{2\alpha+1}$

How Can I Use These Algorithms?

> *In the future, we foresee that the use of area integrals when simulating strong solutions to SDEs will become as automatic as the use of random numbers from a normal distribution is today. After all, once a good routine has been developed and implemented in numerical libraries, the ordinary user will only need to call this routine from each program and will not need to be concerned with the details of how the routine works.*
>
> — *J. G. Gaines and T. J. Lyons (1994)*

How Can I Use These Algorithms?

LevyArea.jl

- fast implementations in Julia
- easy to use
- flexible (different error criteria, correlated noise)
- employs state of the art estimates
- over 60.000 downloads
- MATLAB version also available

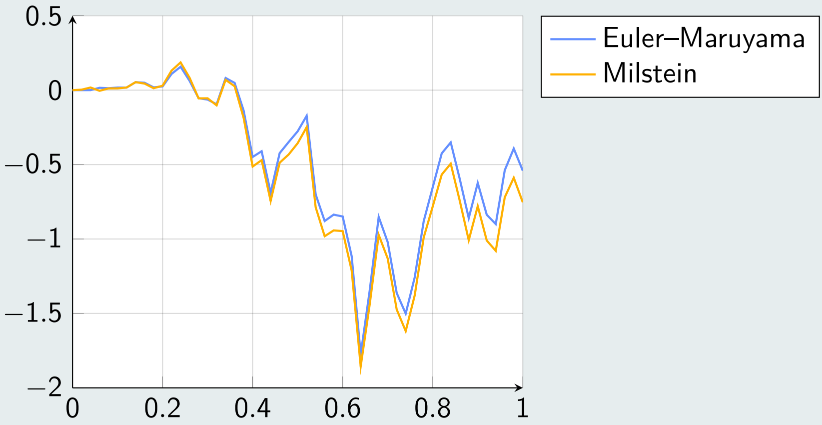
Euler–Maruyama

```
for n = 1:N
    ΔW = sqrt(Δt)*randn(2)


    Y[1,n+1] = Y[1,n] + ΔW[1]
    Y[2,n+1] = Y[2,n] + Y[1,n]*ΔW[2]
end
```

Milstein

```
alg = optimal_algorithm(2, Δt)
for n = 1:N
    ΔW = sqrt(Δt)*randn(2)
    II = iterated_integrals(ΔW, Δt; alg=alg)

    Z[1,n+1] = Z[1,n] + ΔW[1]
    Z[2,n+1] = Z[2,n] + Z[1,n]*ΔW[2] + II[1,2]
end
```

Result



TL;DL

1. What were we talking about?
   - twice iterated stochastic integrals

2. Why do we care?
   - stochastic Taylor expansions
   - approximation of SDEs

3. How do we do it?
   - use `LevyArea.jl`

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